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TOPICAL REVIEW

Quantum dynamics of two-spin-qubit systems

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Abstract

The aim of this topical review is a systematic and concise presentation of the results of a series of theoretical works on the quantum dynamics of two-spin-qubit systems towards the elaboration of a physical mechanism of the quantum information transfer between two spin-qubits. For this purpose the main attention is paid to exactly solvable models of two-spin-qubit systems, since the analytical expressions of the elements of their reduced density matrices explicitly exhibit the mutual dependence of the quantum information encoded into the spin-qubits. The treatment of their decoherence due to the interaction with the environment is performed in the Markovian approximation. Rate equations for axially symmetric systems of two coupled spin-qubits non-interacting, as well as interacting, with the environment are exactly solved. It is shown how the solutions of rate equations demonstrate the physical mechanism of the quantum information exchange between the spin-qubits. This mechanism holds also in all two-spin-qubit systems whose rate equations can be solved only by means of numerical calculations. Exact solutions of rate equations for two uncoupled spin-qubits interacting with two separate environments reveal an interesting physical phenomenon in the time evolution of the qubit–qubit entanglement generated by their interaction with the environments: the entanglement sudden death and revival. A two-spin-qubit system with an asymptotically decoherence free subspace was also explicitly constructed. The presented calculations and reasonings can be extended for application to the study of spin-qubit chains or networks.

Contents

1. Introduction

A basic element of devices and systems for processing quantum information (QI) is the quantum bit or qubit for short [1]. Each qubit is a two-state quantum system. It is usually also called a two-level system with the tacit understanding that both levels are non-degenerate. The QI encoded into a qubit is its two-component wavefunction (for a pure state) or its 2×2 density matrix (for a mixed state). The simplest model of the QI transfer from a qubit to another one is that inside a two-qubit system—a double qubit [1, 2]. The present paper is a review of theoretical works devoted to the study of systems of two interacting qubits for elaborating the physical basis of the QI transfer between them.

There are many types of two-level quantum systems with different physical structures: two energy levels of a spin $1/2$ magnetic particle in a constant magnetic field, a two-level atom, the lowest energy exciton state and the ground state in a semiconductor quantum dot etc. Each spin $1/2$ magnetic particle is called a spin-qubit. For definiteness we chose to study two-spin-qubit systems [3–19]. However, there are many other systems of two interacting qubits whose total Hamiltonian can be represented in the same form as that of a two-spin-qubit system [20–26]. For all two-qubit systems of this kind, the reasonings concerning equivalent two-spin-qubit systems can be applied directly.

The interactions between two spin-qubits may have different mechanisms: direct spin–spin couplings (Heisenberg magnetic exchange interactions, magnetic dipole–dipole interactions, etc), interactions through the intermediary of a spin chain, a spin lattice or other spin systems, and also the effective spin–spin coupling in electrostatically coupled quantum dots [4]. We review the results of the studies on systems of two spin-qubits with their spin–spin couplings. Besides this strong coupling there also exists a weak interaction between qubits and the environment which causes energy dissipation and the dephasing of their coherent oscillations [5, 6, 11–13, 16, 26–35]. Therefore, the total physical system consists of two interacting subsystems, one of them is a two-spin-qubit subsystem and the other subsystem is the environment.

It is natural and convenient to use each product of two basis vectors of the Hilbert spaces of two above-mentioned subsystems as a basis vector of the Hilbert space of the total physical system. The trace of its density matrix ρ_{tot} over the pair of indices labeling state vectors of the environment is a 4×4 matrix ρ called the reduced density matrix, describing the time evolution of the two-spin-qubit subsystem in the presence of its interaction with the environment. From the von Neumann equation for ρ_{tot} there follows a system of integro-differential rate equations (called also the master equations), each of which expresses a time derivative of a matrix element of ρ at a time $t > 0$ in the form of the sum of a linear combination of matrix elements of ρ at the same time t and an integral of another linear combination of elements of ρ with respect to the time variable in the interval from 0 to t: the quantum dynamical equations for the reduced density matrix is non-Markovian [35–43]. However, if the interaction of spin-qubits with the environment is weak and the variation of the physical fields in the environment is slow, then one can assume the Markovian approximation [44, 45]: replace values of matrix elements of ρ in the integral from 0 to t by their values at t and subsequently extend the integral to infinity. In this approximation rate equations become linear differential equations for the elements of the reduced density matrix ρ. The present review is limited to this approximation.

In section 2 the general theories of the decoherence of two-spin-qubit systems in the Markovian approximation [44, 45], as well as in the Born–Markov approximation [46–48] as a special case of the Markovian one, are presented. The systems of rate equations for elements of reduced density matrices ρ in these approximations are derived. In section 3 the solutions of rate equations for systems of two coupled spin-qubits without an interaction with the environment are derived and the physical interpretation of the expressions of these solutions is proposed. Section 4 is devoted to the study of two uncoupled spin-qubits interacting with the environment. Due to this interaction, the environment mediates an indirect effective interaction between two spin-qubits which may cause an interesting physical phenomenon: the entanglement sudden death and revival. Solutions of rate equations for several special systems of two coupled spin-qubits interacting with the environment are established in section 5. Section 6 is the conclusion. Throughout this work the unit system with $\hbar = c = 1$ is used.

2. Rate equations in Markovian approximation

The exact quantum dynamics of a two-spin-qubit system interacting with the environment is governed by the von Neumann equation for the total density matrix ρ_{tot} of the complex system consisting of two spin-qubits and the environment

$$\mathrm{i}\frac{\mathrm{d}\rho_{\text{tot}}}{\mathrm{d}t} = [H_{\text{tot}}, \rho_{\text{tot}}], \tag{1}$$

where H_{tot} is its total Hamiltonian

$$H_{\text{tot}} = H + H_{\text{E}} + H_{\text{int}}, \tag{2}$$

H and H_{E} are Hamiltonians of two separate subsystems: those of two-spin-qubits and the environment, respectively, H_{int} is the Hamiltonian of their interaction. The quantum dynamics of the two-spin-qubit subsystem, also called two-spin-qubit system in the sequel, in the presence of its interaction with the environment is described by the reduced density matrix ρ. It is a 4×4 matrix and can be represented as a linear combination of 15 generators Γ_A of the $SU(4)$ group

$$\rho = \tfrac{1}{4} + \sum_A \Gamma_A \rho_A. \tag{3}$$

In the Markovian approximation from equation (1) one can derive the following equation for the reduced density matrix of the two-spin-qubit system

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\mathrm{i}[H, \rho] + L\rho \tag{4}$$

with some linear operator L called the Liouvillian superoperator. This linear operator consists of two parts:

$$L\rho = L^{(1)}\rho + L^{(2)}\rho, \tag{5}$$

one of which, $L^{(1)}\rho$, is the effect of the renormalization of energy levels leading to the frequency shifts (Lamb shifts) and the other part, $L^{(2)}\rho$, is a completely positive operator describing the dissipative and dephasing actions of the environment. $L^{(1)}\rho$ can be expressed as the addition of a new term δH to the Hamiltonian

$$L^{(1)}\rho = -\mathrm{i}[\delta H, \rho]. \tag{6}$$

δH has following general form

$$\delta H = \tfrac{1}{2}\sum_A \Gamma_A h_A \tag{7}$$

with real constants h_A. Gorini *et al* [49] have rigorously derived the general formula for $L^{(2)}\rho$:

$$L^{(2)}\rho = \tfrac{1}{2}\sum_{AB} \xi_{AB}\{[\Gamma_A, \rho\Gamma_B^+] + [\Gamma_A\rho, \Gamma_B^+]\}, \tag{8}$$

where ξ_{AB} are the elements of a 15×15 positive matrix,

$$\xi_{AB} = \xi^s_{AB} + \mathrm{i}\xi^a_{AB}, \qquad \xi^s_{AB} = \xi^s_{BA}, \qquad \xi^a_{AB} = -\xi^a_{BA}, \tag{9}$$

ξ^s_{AB} and ξ^a_{AB} being real constants. A special case of equation (8) is the Linblad formula [50]

$$L^{(2)}\rho = \tfrac{1}{2}\sum_{A} \xi_{A}\{[\Gamma_A, \rho\Gamma_A^+] + [\Gamma_A\rho, \Gamma_A^+]\}, \tag{10}$$

which was often used for the study of the quantum systems with decoherence.

The Hamiltonian of the interaction of qubits with the environment $H_{\rm int}$ can be represented in the form

$$H_{\rm int} = \sum_{\mu\nu} |\mu\rangle K_{\mu\nu}\langle\nu|, \tag{11}$$

where operators $K_{\mu\nu}$ in the Hilbert space of the state vectors of the environment are expressed in terms of quantum operators of the environment in the Schrödinger picture. We chose $K_{\mu\nu}$ to have a vanishing statistical average over all the states of the environment at a given temperature T

$$\langle K_{\mu\nu}\rangle_\beta = 0, \tag{12}$$

where $\beta = kT$, k is the Boltzmann constant, and $\langle\cdots\rangle_\beta$ denotes the statistical average. In the second (lowest) order of the perturbation theory with respect to interaction Hamiltonian (11)—the Born–Markov approximation, the explicit form of Liouvillian superoperator $L\rho$ can be established in the framework of the Bloch–Redfield formalism [46–48]. Its matrix elements $(L\rho)_{\mu\nu}$,

$$L\rho = \sum_{\mu\nu} |\mu\rangle (L\rho)_{\mu\nu}\langle\nu|, \tag{13}$$

are expressed by the Redfield formula

$$(L\rho)_{\mu\nu} = -\sum_{\sigma\tau} R_{\mu\nu\sigma\tau}\rho_{\sigma\tau} \tag{14}$$

with the Redfield tensor determined by following relations

$$R_{\mu\nu\sigma\tau} = \delta_{\nu\tau}\sum_\lambda \Gamma^{(+)}_{\mu\lambda\lambda\sigma} + \delta_{\mu\sigma}\sum_\lambda \Gamma^{(-)}_{\tau\lambda\lambda\nu} - \Gamma^{(+)}_{\tau\nu\mu\sigma} - \Gamma^{(-)}_{\tau\nu\mu\sigma}, \tag{15}$$

$$\begin{aligned} \Gamma^{(+)}_{\mu\sigma\nu\tau} &= \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}\omega_{\nu\tau}t}\langle K_{\mu\sigma}(t)K_{\nu\tau}(0)\rangle_\beta, \\ \Gamma^{(-)}_{\mu\sigma\nu\tau} &= \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}\omega_{\mu\sigma}t}\langle K_{\mu\sigma}(0)K_{\nu\tau}(t)\rangle_\beta, \end{aligned} \tag{16}$$

where $\omega_{\mu\sigma}$ is the difference of the energies of states 'μ' and 'σ', $\omega_{\mu\sigma} = E_\mu - E_\sigma$, $K_{\mu\sigma}(t)$ is operator $K_{\mu\sigma}$ in the interaction picture

$$K_{\mu\sigma}(t) = \mathrm{e}^{\mathrm{i}H_{\rm E}t}K_{\mu\sigma}\mathrm{e}^{-\mathrm{i}H_{\rm E}t}. \tag{17}$$

Since the interaction Hamiltonian is hermitian, operators $K_{\mu\sigma}$ must satisfy the condition

$$K^+_{\sigma\mu} = K_{\mu\sigma}. \tag{18}$$

From this condition it is straightforward to derive a relation for the matrix elements (16):

$$(\Gamma^{(+)}_{\mu\sigma\nu\tau})^* = \Gamma^{(-)}_{\tau\nu\sigma\mu}, \tag{19}$$

and also following hermiticity property of Redfield tensor:

$$\left(R_{\nu\mu\sigma\tau}\right)^* = R_{\mu\nu\tau\sigma}. \tag{20}$$

For studying rate equations it is convenient to use hermitian generators Γ_A

$$\Gamma^+_A = \Gamma_A. \tag{21}$$

We chose them to satisfy condition

$$\mathrm{Tr}(\Gamma_A\Gamma_B) = 4\delta_{AB} \tag{22}$$

and denote by f_{ABC} the corresponding structure constants of the $SU(4)$ group

$$[\Gamma_A, \Gamma_B] = 2\mathrm{i}\sum_C f_{ABC}\Gamma_C. \tag{23}$$

Matrices $L\rho$ and $L^{(i)}\rho$ are expressed in terms of Γ_A by formulae similar to equation (3)

$$L\rho = \sum_A \Gamma_A(L\rho)_A, \qquad L^{(i)}\rho = \sum_A \Gamma_A(L^{(i)}\rho)_A. \tag{24}$$

Then the general formulae (5)–(8) can be rewritten in the new forms

$$(L\rho)_A = (L^{(1)}\rho)_A + (L^{(2)}\rho)_A, \tag{25}$$

$$(L^{(1)}\rho)_A = \sum_{BC} f_{ACB}h_C\rho_B, \tag{26}$$

$$(L^{(2)}\rho)_A = -\tilde\lambda_A - \sum_B \tilde\lambda_{AB}\rho_B, \tag{27}$$

where

$$\tilde\lambda_A = -\tfrac{1}{16}\sum_{CD}\xi_{CD}\mathrm{Tr}\{\Gamma_A[\Gamma_C, \Gamma_D]\}, \tag{28}$$

$$\tilde\lambda_{AB} = -\tfrac{1}{8}\sum_{CD}\xi_{CD}\mathrm{Tr}\{\Gamma_A[\Gamma_C\Gamma_B, \Gamma_D] + \Gamma_A[\Gamma_C, \Gamma_B\Gamma_D]\}. \tag{29}$$

Note that if ξ_{CD} is a symmetric tensor, then

$$\tilde{\lambda}_A = 0, \qquad \tilde{\lambda}_{AB} = \tilde{\lambda}_{BA}. \tag{30}$$

On the other hand, from equation (14) there follows a relation between $(L\rho)_A$ and ρ_A in the Bloch–Redfield formalism

$$(L\rho)_A = -\lambda_A - \sum_B \lambda_{AB}\rho_B, \tag{31}$$

where

$$\lambda_A = \tfrac{1}{16}\sum_{\mu\nu\sigma} (\Gamma_A)_{\nu\mu} R_{\mu\nu\sigma\sigma}, \tag{32}$$

$$\lambda_{AB} = \tfrac{1}{4}\sum_{\mu\nu\sigma\tau} (\Gamma_A)_{\nu\mu} R_{\mu\nu\sigma\tau} (\Gamma_B)_{\sigma\tau}. \tag{33}$$

From the hermiticity property (20) of the Redfield tensor it follows that λ_A and λ_{AB} are real constants. Formulae (31)–(33) of the Bloch–Redfield formalism must be compatible with those derived without using the perturbation theory, equations (25)–(29). Therefore we have the following equations

$$\lambda_A = \tilde{\lambda}_A, \tag{34}$$

$$\lambda_{AB} = \sum_C f_{ABC} h_C + \tilde{\lambda}_{AB}, \tag{35}$$

where $\lambda_A, \tilde{\lambda}_A$ and $\lambda_{AB}, \tilde{\lambda}_{AB}$ are determined by formulae (32), (28) and (33), (29), respectively. By solving these systems of equations for constants h_C and coefficients ξ_{CD}, we obtain their expressions in the second order approximation of the perturbation theory in terms of components of the Redfield tensor.

In order to derive rate equations from equation (4) we write Hamiltonian H in the general form

$$H = \tfrac{1}{2}\sum_A \Gamma_A E_A. \tag{36}$$

Then we obtain the following system of 15 differential equations for the 15 components ρ_A:

$$\frac{\mathrm{d}\rho_A}{\mathrm{d}t} = \sum_{BC} f_{ABC} E_B \rho_C - \lambda_A - \sum_B \lambda_{AB}\rho_B. \tag{37}$$

It is worth noting that the results of this section with a slight modification can be applied to the study of the decoherence of any N-level system.

3. Two coupled spin-qubits without interaction with the environment

Let us study special cases in which the interaction of the environment on the qubits can be neglected and, therefore, the reduced density matrix ρ mentioned in section 2 is the density matrix itself of the closed system of two coupled spin-qubits. Its time evolution is determined by the von Neumann equation

$$\mathrm{i}\frac{\mathrm{d}\rho}{\mathrm{d}t} = [H, \rho]. \tag{38}$$

Introducing the Pauli matrices σ_α or τ_α, $\alpha = x, y, z$ or $+, -, 3$, and unit matrices $\sigma_0 = 1$, $\tau_0 = 1$, acting on the spin indices of the wave functions of the corresponding spin-qubits called the σ-qubit and τ-qubit, and denoting by ρ^σ or ρ^τ the trace of ρ over the spin indices of the τ-qubit and σ-qubit, respectively,

$$\rho^\sigma = \mathrm{Tr}_\tau \rho, \qquad \rho^\tau = \mathrm{Tr}_\sigma \rho. \tag{39}$$

Being density matrices of two spin-qubits, ρ^σ and ρ^τ have the general form

$$\rho^\sigma = \tfrac{1}{2} + \sum_\alpha \sigma_\alpha S_\alpha(t), \qquad \rho^\tau = \tfrac{1}{2} + \sum_\alpha \tau_\alpha T_\alpha(t). \tag{40}$$

Suppose that at the initial time $t = 0$ the quantum states of the two spin-qubits are independent and therefore

$$\rho_{(\alpha\beta)}(0) = S_\alpha(0) T_\beta(0), \qquad \alpha \neq 0, \quad \beta \neq 0. \tag{41}$$

3.1. Exactly solvable model

Consider the model of two identical spin 1/2 magnetic particles with their anisotropic axially symmetric Heisenberg exchange interaction in a constant homogeneous perpendicular magnetic field. Hamiltonian H has a simple expression

$$H = \frac{E}{2}(\sigma_3 + \tau_3) + \frac{1}{2}J_\parallel \sigma_3\tau_3 + J_\perp(\sigma_+\tau_- + \sigma_-\tau_+), \tag{42}$$

where E is the energy difference of the two levels and $J_\parallel$, $J_\perp$ are the coupling constants. In this case equation (38) can be solved exactly. In terms of the components $S_\alpha(t)$ and $T_\alpha(t)$ in the rhs of equation (40), $\alpha = 1, 2, 3$ or x, y, z, its solution is represented by the following expressions:

$$\begin{aligned}S_x(t) = {} & \tfrac{1}{2}(\cos J^{(-)}t + \cos J^{(+)}t)[\cos Et S_x(0) - \sin Et S_y(0)] \\ & + \tfrac{1}{2}(\cos J^{(-)}t - \cos J^{(+)}t)[\cos Et T_x(0) - \sin Et T_y(0)] \\ & - (\sin J^{(-)}t + \sin J^{(+)}t) T_3(0)[\sin Et S_x(0) \\ & + \cos Et S_y(0)] - (\sin J^{(-)}t - \sin J^{(+)}t) S_3(0) \\ & \times [\sin Et T_x(0) + \cos Et T_y(0)],\end{aligned} \tag{43}$$

$$\begin{aligned}S_y(t) = {} & \tfrac{1}{2}(\cos J^{(-)}t + \cos J^{(+)}t)[\sin Et S_x(0) + \cos Et S_y(0)] \\ & + \tfrac{1}{2}(\cos J^{(-)}t - \cos J^{(+)}t)[\sin Et T_x(0) + \cos Et T_y(0)] \\ & + (\sin J^{(-)}t + \sin J^{(+)}t) T_3(0)[\cos Et S_x(0) \\ & - \sin Et S_y(0)] + (\sin J^{(-)}t - \sin J^{(+)}t) S_3(0) \\ & \times [\cos Et T_x(0) - \sin Et T_y(0)],\end{aligned} \tag{44}$$

$$\begin{aligned}S_z(t) = {} & \tfrac{1}{2}(1 + \cos 2J_\perp t) S_z(0) + \tfrac{1}{2}(1 - \cos 2J_\perp t) T_z(0) \\ & - \sin 2J_\perp t [S_x(0) T_y(0) - S_y(0) T_x(0)],\end{aligned} \tag{45}$$

where

$$J^{(\pm)} = J_\parallel \pm J_\perp. \tag{46}$$

For $T_x(t)$, $T_y(t)$ and $T_z(t)$ we have similar formulae with the interchange of the initial values $S_\alpha(0) \leftrightarrow T_\alpha(0)$, $\alpha = x, y, z$. In terms of expressions of $S_\alpha(t)$ and $T_\alpha(t)$ for free qubits,

$$\begin{aligned}S_x^{\mathrm{free}}(t) &= \cos Et S_\alpha(0) - \sin Et S_y(0), \\ S_y^{\mathrm{free}}(t) &= \sin Et S_\alpha(0) + \cos Et S_y(0), \\ S_z^{\mathrm{free}}(t) &= S_z(0)\end{aligned} \tag{47}$$

and similarly for $T_x^{\rm free}$, $T_y^{\rm free}$, $T_z^{\rm free}$, equations (43)–(45) can be rewritten in the forms

$$\begin{aligned} S_x(t) &= \tfrac{1}{2}(\cos J^{(-)}t + \cos J^{(+)}t)S_x^{\rm free}(t) \\ &\quad + \tfrac{1}{2}(\cos J^{(-)}t - \cos J^{(+)}t)T_x^{\rm free}(t) \\ &\quad - (\sin J^{(-)}t + \sin J^{(+)}t)T_3(0)S_y^{\rm free}(t) \\ &\quad - (\sin J^{(-)}t - \sin J^{(+)}t)S_3(0)T_y^{\rm free}(t), \end{aligned} \tag{48}$$

$$\begin{aligned} S_y(t) &= \tfrac{1}{2}(\cos J^{(-)}t + \cos J^{(+)}t)S_y^{\rm free}(t) \\ &\quad + \tfrac{1}{2}(\cos J^{(-)}t - \cos J^{(+)}t)T_y^{\rm free}(t) \\ &\quad + (\sin J^{(-)}t + \sin J^{(+)}t)T_3(0)S_x^{\rm free}(t) \\ &\quad + (\sin J^{(-)}t - \sin J^{(+)}t)S_3(0)T_x^{\rm free}(t), \end{aligned} \tag{49}$$

$$\begin{aligned} S_z(t) &= \tfrac{1}{2}(1 + \cos 2J_\perp t)S_z^{\rm free}(t) + \tfrac{1}{2}(1 - \cos 2J_\perp t)T_z^{\rm free}(t) \\ &\quad - \sin 2J_\perp t[S_x(0)T_y(0) - S_y(0)T_x(0)], \end{aligned} \tag{50}$$

and similarly for $T_x(t)$, $T_y(t)$, $T_z(t)$.

Formulae (43)–(45) show that the output QI from the σ-qubit $S_x(t)$, $S_y(t)$, $S_z(t)$ at $t > 0$ reflects the presence of the τ-qubit and also the input QI encoded into it at the initial time $t = 0$. This phenomenon can be considered as a physical mechanism of the QI transfer from the τ-qubit to the σ-qubit. Similarly, formulae for $T_x(t)$, $T_y(t)$, $T_z(t)$ demonstrate that of the QI transfer from the σ-qubit to the τ-qubit.

For studying entanglement of two qubits it is convenient to use collective Dicke states [51]

$$|e\rangle = |11\rangle, \qquad |g\rangle = |22\rangle,$$
$$|s\rangle = \frac{1}{\sqrt{2}}[|21\rangle + |12\rangle], \qquad |a\rangle = \frac{1}{\sqrt{2}}[|21\rangle - |12\rangle] \tag{51}$$

as the basis vectors. In this basis, matrix elements of ρ have simple expressions

$$\begin{aligned} \rho_{ee}(t) &= \rho_{ee}(0), & \rho_{gg}(t) &= \rho_{gg}(0), \\ \rho_{ss}(t) &= \rho_{ss}(0), & \rho_{aa}(t) &= \rho_{aa}(0), \\ \rho_{sa}(t) &= {\rm e}^{-2{\rm i}J_\perp t}\rho_{sa}(0), & \rho_{as}(t) &= {\rm e}^{2{\rm i}J_\perp t}\rho_{as}(0), \\ \rho_{eg}(t) &= {\rm e}^{-2{\rm i}Et}\rho_{eg}(0), & \rho_{ge}(t) &= {\rm e}^{2{\rm i}Et}\rho_{ge}(0), \\ \rho_{es}(t) &= {\rm e}^{-{\rm i}(E+J^{(-)})t}\rho_{es}(0), & \rho_{se}(t) &= {\rm e}^{{\rm i}(E+J^{(-)})t}\rho_{se}(0), \\ \rho_{gs}(t) &= {\rm e}^{-{\rm i}(E-J^{(-)})t}\rho_{gs}(0), & \rho_{sg}(t) &= {\rm e}^{{\rm i}(E-J^{(-)})t}\rho_{sg}(0), \\ \rho_{ea}(t) &= {\rm e}^{-{\rm i}(E+J^{(+)})t}\rho_{ea}(0), & \rho_{ae}(t) &= {\rm e}^{{\rm i}(E+J^{(+)})t}\rho_{ae}(0), \\ \rho_{ga}(t) &= {\rm e}^{{\rm i}(E-J^{(+)})t}\rho_{ga}(0), & \rho_{ag}(t) &= {\rm e}^{-{\rm i}(E-J^{(+)})t}\rho_{ag}(0). \end{aligned} \tag{52}$$

It follows that for six classes of two-qubit quantum states with special initial conditions

class 1:

$$\rho_{es}(0) = \rho_{se}(0) = \rho_{ea}(0) = \rho_{ae}(0) = \rho_{gs}(0) = \rho_{sg}(0) = \rho_{ga}(0) = \rho_{ag}(0) = 0,$$

class 2:

$$\rho_{eg}(0) = \rho_{ge}(0) = \rho_{es}(0) = \rho_{se}(0) = \rho_{ea}(0) = \rho_{ae}(0) = \rho_{gs}(0) = \rho_{sg}(0) = \rho_{ga}(0) = \rho_{ag}(0) = 0,$$

class 3:

$$\rho_{eg}(0) = \rho_{ge}(0) = \rho_{sa}(0) = \rho_{as}(0) = \rho_{es}(0) = \rho_{se}(0) = \rho_{gs}(0) = \rho_{sg}(0) = \rho_{ga}(0) = \rho_{ag}(0) = 0,$$

class 4:

$$\rho_{eg}(0) = \rho_{ge}(0) = \rho_{sa}(0) = \rho_{as}(0) = \rho_{es}(0) = \rho_{se}(0) = \rho_{ea}(0) = \rho_{ae}(0) = \rho_{ga}(0) = \rho_{ag}(0) = 0,$$

class 5:

$$\rho_{eg}(0) = \rho_{ge}(0) = \rho_{sa}(0) = \rho_{as}(0) = \rho_{es}(0) = \rho_{se}(0) = \rho_{ea}(0) = \rho_{ae}(0) = \rho_{gs}(0) = \rho_{sg}(0) = 0,$$

class 6:

$$\rho_{eg}(0) = \rho_{ge}(0) = \rho_{sa}(0) = \rho_{as}(0) = \rho_{es}(0) = \rho_{se}(0) = \rho_{ag}(0) = \rho_{ga}(0),$$
$$\rho_{ea}(0) = \rho_{ae}(0)^*, \qquad \rho_{gs}(0) = \rho_{sg}(0)^*,$$

the concurrence of the two-qubit system is t-independent and, therefore, the degree of the entanglement between two qubits is conserved. The states of class 6 were called 'X' states [30].

3.2. Arbitrary two-spin-qubit system

Different systems of two coupled spin-qubits with complicated total Hamiltonians H were investigated in [3, 4, 7, 8, 10, 12, 13, 16, 20]. To study any of them we use the natural basis $|(i_1, i_2)\rangle$ of the Hilbert space of state vectors of two spin 1/2 particles,

$$\begin{aligned} \sigma_3|(1, i_2)\rangle &= |(1, i_2)\rangle, & \sigma_3|(2, i_2)\rangle &= -|(2, i_2)\rangle, \\ \tau_3|(i_1, 1)\rangle &= |(i_1, 1)\rangle, & \tau_3|(i_1, 2)\rangle &= -|(i_1, 2)\rangle, \end{aligned} \tag{53}$$

and introduce a double index I to replace the pair $(i_1 i_2)$: $(i_1 i_2) \rightarrow I$.

Denote by $|\psi_\mu\rangle$ the eigenstates of the total Hamiltonian corresponding to the eigenvalues E_μ,

$$H|\psi_\mu\rangle = E_\mu|\psi_\mu\rangle, \tag{54}$$

and expand them in terms of basis vectors

$$|\psi_\mu\rangle = \sum_{i_1 i_2} C_\mu^{(i_1 i_2)}\,|(i_1 i_2)\rangle = \sum_I C_\mu^I|I\rangle. \tag{55}$$

The 4×4 matrix with elements C_μ^I is unitary

$$\sum_\mu (C_\mu^I)^*(C_\mu^J) = \delta_{IJ}, \sum_I (C_\mu^I)^*(C_\nu^I) = \delta_{\mu\nu}, \tag{56}$$

and expansion (55) has following inversion

$$|I\rangle = \sum_\mu (C_\mu^I)^*|\psi_\mu\rangle. \tag{57}$$

Consider an arbitrary two-spin-qubit system with a direct spin–spin coupling and without interactions with the environment. Denote by ρ_{IJ} the elements of density matrix ρ in the natural basis and by $\rho_{\mu\nu}$ those in the basis of eigenstates $|\psi_\mu\rangle$ of total Hamiltonian:

$$\rho_{IJ}(t) = \langle I|\rho(t)|J\rangle, \qquad \rho_{\mu\nu} = \langle\psi_\mu|\rho(t)|\psi_\nu\rangle. \tag{58}$$

Between the two systems of matrix elements there exist the following relations

$$\rho_{IJ}(t) = \sum_{\mu\nu} C^I_\mu \rho_{\mu\nu}(t)(C^J_\nu)^*, \tag{59}$$

$$\rho_{\mu\nu}(t) = \sum_{IJ} (C^I_\mu)^* \rho_{IJ}(t) C^J_\nu. \tag{60}$$

Since $|\psi_\mu\rangle$ are eigenvectors of H, each matrix element $\rho_{\mu\nu}(t)$ has a simple t-dependence

$$\rho_{\mu\nu}(t) = \mathrm{e}^{-\mathrm{i}(E_\mu - E_\nu)t} \rho_{\mu\nu}(0). \tag{61}$$

From equations (59)–(61) it follows that

$$\rho_{IJ}(t) = \sum_{\mu\nu}\sum_{KL} \mathrm{e}^{-\mathrm{i}(E_\mu - E_\nu)t} C^I_\mu (C^J_\nu)^* (C^K_\mu)^* C^L_\nu \rho_{KL}(0), \tag{62}$$

or in the explicit form with pair indices

$$\rho_{(i_1i_2)(j_1j_2)}(t) = \sum_{\mu\nu}\sum_{k_1k_2}\sum_{l_1l_2} \mathrm{e}^{-\mathrm{i}(E_\mu - E_\nu)t} C^{(i_1i_2)}_\mu (C^{(j_1j_2)}_\nu)^* \times (C^{(k_1k_2)}_\mu)^* C^{(l_1l_2)}_\nu \rho_{(k_1k_2)(l_1l_2)}(0). \tag{63}$$

Matrix elements $\rho_{(i_1i_2)(j_1j_2)}(t)$ are expressed in terms of components $\rho_{(\alpha\beta)}(t), (\alpha\beta) \neq (00), \alpha, \beta = 0, x, y, z$ or, equivalently, $\alpha, \beta = 0, 1, 2, 3$,

$$\rho_{(i_1i_2)(j_1j_2)}(t) = \tfrac{1}{4}\delta_{i_1j_1}\delta_{i_2j_2} + \sum_{(\alpha\beta)\neq(00)} (\sigma_\alpha)_{i_1j_1}(\tau_\beta)_{i_2j_2}\rho_{(\alpha\beta)}(t). \tag{64}$$

Inversely we have

$$\rho_{(\alpha\beta)}(t) = \tfrac{1}{4}\sum_{i_1i_2}\sum_{j_1j_2} (\sigma_\alpha)_{j_1i_1}(\tau_\beta)_{j_2i_2}\rho_{(i_1i_2)(j_1j_2)}(t). \tag{65}$$

Using this expression of $\rho_{(\alpha\beta)}(t)$ in terms of the matrix elements, formula (62) for the time evolution of the matrix elements and formula (64) expressing the matrix elements in terms of the components at $t = 0$, we derive a relation determining the time evolution of components $\rho_{(\alpha\beta)}(t)$:

$$\rho_{(\alpha\beta)}(t) = \tfrac{1}{4}\sum_{i_1i_2}\sum_{j_1j_2} (\sigma_\alpha)_{j_1i_1}(\tau_\beta)_{j_2i_2} \cdot \sum_{\mu\nu}\sum_{k_1k_2}\sum_{l_1l_2} \mathrm{e}^{-\mathrm{i}(E_\mu - E_\nu)t} \times C^{(i_1i_2)}_\mu (C^{(j_1j_2)}_\nu)^* (C^{(k_1k_2)}_\mu)^* C^{(l_1l_2)}_\nu \times \left[\tfrac{1}{4}\delta_{k_1l_1}\delta_{k_2l_2} + \sum_{\gamma\delta} (\sigma_\gamma)_{k_1l_1}(\tau_\delta)_{k_2l_2}\rho_{\gamma\delta}(0)\right]. \tag{66}$$

For each concrete system with a given total Hamiltonian H, eigenvalues E_μ and coefficients $C^{(i_1i_2)}_\mu$ in formula (55) of eigenstates can be calculated by solving the Schrödinger equation (54). Substituting their values into the rhs of equation (66), we derive expressions of components $\rho_{(\alpha\beta)}(t)$ containing different numerical coefficients. As particular cases we obtain formulae for components $S_\alpha(t)$ and $T_\alpha(t)$ determining reduced density matrices of two spin-qubits. These formulae demonstrate the physical mechanism of the QI exchange between two qubits.

Each system with a total Hamiltonian in the form (42) is a special case when eigenvalues E_μ and coefficients C^I_μ are analytical expressions of its physical parameters. In general, eigenvalues E_μ can be determined by means of numerical calculations and all coefficients C^I_μ are expressed in terms of them and physical parameters of the system. The problem is almost exactly solvable. For example, among four eigenvalues E_μ of a total Hamiltonian of the form [7, 12, 13, 20]

$$H = \frac{E}{2}(\sigma_3 + \tau_3) + \frac{\Delta}{2}(\sigma_1 + \tau_1) + \frac{J_\parallel}{2}\sigma_3\tau_3 + J_\perp(\sigma_+\tau_- + \sigma_-\tau_+), \tag{67}$$

there is only one exactly determined eigenvalue $E_0 = -(J_\perp + J_\parallel/2)$, the three others $E_\mu, \mu = 1, 2, 3$, are the roots of a rank three algebraic equation

$$\left(x - \frac{J_\parallel}{2}\right)^2 \left(x - J_\perp + \frac{J_\parallel}{2}\right) - E^2\left(x - J_\perp + \frac{J_\parallel}{2}\right) - \Delta^2\left(x - \frac{J_\parallel}{2}\right) = 0. \tag{68}$$

Coefficients C^I_μ are expressed in terms of the four eigenvalues E_μ and the physical parameters $E, \Delta, J_\parallel, J_\perp$ of the system and can be calculated numerically.

4. Influence of decoherence on qubit–qubit entanglement

In solids there always exists the interaction of qubits with the environment. It may generate decoherence of the qubits, induce an effective interaction between two qubits and/or affect their entanglement. In order to study separately the influence of the dissipative interaction of the environment on the qubit–qubit entanglement let us consider a system of two uncoupled spin-qubits interacting with the environment. The reduced density matrix ρ of the system of two uncoupled spin-qubits can be determined by solving a system of rate equations of the form (37) with vanishing components $E_{(\alpha\beta)}, \alpha \neq 0, \beta \neq 0$. Usually the axially symmetrical system of two identical spin-qubits with $E_{(\pm 0)} = E_{(0\pm)} = 0, E_{(30)} = E_{(03)} = E$ is considered.

The simplest example is the case of the two above-mentioned identical non-coupled spin-qubits interacting with two separate noise environments [26, 29, 30, 32, 33, 35, 36] with the interaction Hamiltonian of the form

$$H_{\mathrm{int}} = \sum_\xi [f^*_\xi(\sigma_- a^{\sigma+}_\xi + \tau_- a^{\tau+}_\xi) + f_\xi(\sigma_+ a^\sigma_\xi + \tau_+ a^\tau_\xi)] + \sum_\xi [\sigma_3(g^*_\xi b^{\sigma+}_\xi + g_\xi b^\sigma_\xi) + \tau_3(g^*_\xi b^{\tau+}_\xi + g_\xi b^\tau_\xi)], \tag{69}$$

where $a^\sigma_\xi, b^\sigma_\xi$ and $a^{\sigma+}_\xi, b^{\sigma+}_\xi$ are the destruction and creation operators for the bosons in the two noise sources of the σ-qubit, a^τ_ξ, b^τ_ξ and $a^{\tau+}_\xi, b^{\tau+}_\xi$ are similar operators for the τ-qubit. Using formula (69) for the interaction Hamiltonian and equations (14)–(17) we derive the expression of the Liouvillian superoperator in the Born–Markov approximation

$$L^{(2)}\rho = \tfrac{1}{2}\sum_{\alpha,\beta=x,y,z} c_{\alpha\beta}\{[\sigma_\alpha, \rho\sigma_\beta] + [\sigma_\alpha\rho, \sigma_\beta] + [\tau_\alpha, \rho\tau_\beta] + [\tau_\alpha\rho, \tau_\beta]\} \tag{70}$$

with the following non-vanishing constants $c_{\alpha\beta}$:

$$c_{xx} = c_{yy} = \frac{\pi}{2}\sum_{\xi}(1+2n_{\xi})|f_{\xi}|^2\delta(\omega_{\xi}^{a}-E),$$
$$c_{xy} = -c_{yx} = \mathrm{i}\frac{\pi}{2}\sum_{\xi}|f_{\xi}|^2\delta(\omega_{\xi}^{a}-E), \quad (71)$$
$$c_{zz} = \frac{\pi}{2}\sum_{\xi}|g_{\xi}|^2\delta(\omega_{\xi}^{b}),$$

where ω_{ξ}^{a} and ω_{ξ}^{b} are the energies of the bosons with the destruction operators $a_{\xi}^{\sigma}, a_{\xi}^{\tau}$ and $b_{\xi}^{\sigma}, b_{\xi}^{\tau}$, respectively, in the two noise sources, n_{ξ} is the value of the boson numbers

$$n_{\xi} = \langle a_{\xi}^{\sigma+}a_{\xi}^{\sigma}\rangle = \langle a_{\xi}^{\tau+}a_{\xi}^{\tau}\rangle \quad (72)$$

in the equilibrium environments at a given temperature. The system of rate equations has an exact solution. To simplify the text we present here formulae of this solution in the special case of the equilibrium environments at zero temperature, $n_{\xi} = 0$. Using the Dicke basis we have the following expressions of the elements of the reduced density matrix ρ of the two-spin-qubit system:

$$\rho_{ee}(t) = \mathrm{e}^{-2\gamma_1 t}\rho_{ee}(0) + \tfrac{1}{2}(1-\mathrm{e}^{-\gamma_1 t}), \quad (73a)$$
$$\rho_{gg}(t) = -\mathrm{e}^{-\gamma_1 t}(1-\mathrm{e}^{-\gamma_1 t})\rho_{ee}(0) + \mathrm{e}^{-\gamma_1 t} + \tfrac{1}{2}(1-\mathrm{e}^{-\gamma_1 t}), \quad (73b)$$
$$\rho_{ss}(t) = \tfrac{1}{2}(\mathrm{e}^{-2\gamma_1 t}+\mathrm{e}^{-2\gamma_2 t})\rho_{ss}(0) + \tfrac{1}{2}(\mathrm{e}^{-2\gamma_1 t}-\mathrm{e}^{-2\gamma_2 t})\rho_{aa}(0) + \tfrac{1}{2}\mathrm{e}^{-\gamma_1 t}(1-\mathrm{e}^{-\gamma_1 t}) \times [\rho_{ee}(0)-\rho_{gg}(0)] + \tfrac{1}{2}\mathrm{e}^{-\gamma_1 t}(1-\mathrm{e}^{-\gamma_1 t}), \quad (73c)$$
$$\rho_{aa}(t) = \tfrac{1}{2}(\mathrm{e}^{-2\gamma_1 t}-\mathrm{e}^{-2\gamma_2 t})\rho_{ss}(0) + \tfrac{1}{2}(\mathrm{e}^{-2\gamma_1 t}+\mathrm{e}^{-2\gamma_2 t})\rho_{aa}(0) + \tfrac{1}{2}\mathrm{e}^{-\gamma_1 t}(1-\mathrm{e}^{-\gamma_1 t}) \times [\rho_{ee}(0)-\rho_{gg}(0)] + \tfrac{1}{2}\mathrm{e}^{-\gamma_1 t}(1-\mathrm{e}^{-\gamma_1 t}), \quad (73d)$$
$$\rho_{eg}(t) = \mathrm{e}^{-2\mathrm{i}Et}\mathrm{e}^{-2\gamma_2 t}\rho_{eg}(0), \quad (73e)$$
$$\rho_{ge}(t) = \mathrm{e}^{2\mathrm{i}Et}\mathrm{e}^{-2\gamma_2 t}\rho_{ge}(0), \quad (73f)$$
$$\rho_{es}(t) = \mathrm{e}^{-\mathrm{i}Et}\mathrm{e}^{-(\gamma_1+\gamma_2)t}\rho_{es}(0), \quad (73g)$$
$$\rho_{se}(t) = \mathrm{e}^{\mathrm{i}Et}\mathrm{e}^{-(\gamma_1+\gamma_2)t}\rho_{se}(0), \quad (73h)$$
$$\rho_{sg}(t) = \mathrm{e}^{-\mathrm{i}Et}\mathrm{e}^{-\gamma_2 t}[\rho_{sg}(0) + (1-\mathrm{e}^{-\gamma_1 t})\rho_{es}(0)], \quad (73i)$$
$$\rho_{gs}(t) = \mathrm{e}^{\mathrm{i}Et}\mathrm{e}^{-\gamma_2 t}[\rho_{gs}(0) + (1-\mathrm{e}^{-\gamma_1 t})\rho_{se}(0)], \quad (73j)$$
$$\rho_{ea}(t) = \mathrm{e}^{-\mathrm{i}Et}\mathrm{e}^{-(\gamma_1+\gamma_2)t}\rho_{ea}(0), \quad (73k)$$
$$\rho_{ae}(t) = \mathrm{e}^{\mathrm{i}Et}\mathrm{e}^{-(\gamma_1+\gamma_2)t}\rho_{ae}(0), \quad (73l)$$
$$\rho_{ag}(t) = \mathrm{e}^{-\mathrm{i}Et}\mathrm{e}^{-\gamma_2 t}[\rho_{ag}(0) - (1-\mathrm{e}^{-\gamma_1 t})\rho_{ea}(0)], \quad (73m)$$
$$\rho_{ga}(t) = \mathrm{e}^{\mathrm{i}Et}\mathrm{e}^{-\gamma_2 t}[\rho_{ga}(0) - (1-\mathrm{e}^{-\gamma_1 t})\rho_{ae}(0)], \quad (73n)$$
$$\rho_{sa}(t) = \tfrac{1}{2}(\mathrm{e}^{-\gamma_1 t}+\mathrm{e}^{-2\gamma_2 t})\rho_{sa}(0) + \tfrac{1}{2}(\mathrm{e}^{-\gamma_1 t}-\mathrm{e}^{-2\gamma_2 t}) \times \rho_{as}(0), \quad (73o)$$
$$\rho_{as}(t) = \tfrac{1}{2}(\mathrm{e}^{-\gamma_1 t}+\mathrm{e}^{-2\gamma_2 t})\rho_{as}(0) + \tfrac{1}{2}(\mathrm{e}^{-\gamma_1 t}-\mathrm{e}^{-2\gamma_2 t}) \times \rho_{sa}(0), \quad (73p)$$

where

$$\gamma_1 = 2\pi\sum_{\xi}|f_{\xi}|^2\delta(\omega_{\xi}^{a}-E), \qquad \gamma_2 = 4\pi\sum_{\xi}|g_{\xi}|^2\delta(\omega_{\xi}^{b}). \quad (74)$$

From expressions (73a)–(73p) of the elements of the reduced density matrix ρ it follows that for the class of quantum states satisfying conditions

$$\rho_{es} = \rho_{se} = \rho_{gs} = \rho_{sg} = \rho_{ea} = \rho_{ae} = \rho_{ga} = \rho_{ag} = \rho_{as} = \rho_{sa} = 0, \quad (75)$$
$$\rho_{ge} = \rho_{eg}^{*}$$

at $t = 0$, these conditions still hold at any $t > 0$. The concurrence has a simple expression

$$C(t) = \max\{0, C_1(t), C_2(t)\} \quad (76)$$

with

$$C_1(t) = 2|\rho_{ge}(t)| - [\rho_{aa}(t) + \rho_{ss}(t)],$$
$$C_2(t) = |\rho_{ss}(t) - \rho_{aa}(t)| - 2\sqrt{\rho_{gg}(t)\rho_{ee}(t)}. \quad (77)$$

Analogously, for a class of 'X'-states satisfying conditions

$$\rho_{eg} = \rho_{ge} = \rho_{es} = \rho_{se} = \rho_{ag} = \rho_{ga} = \rho_{as} = \rho_{sa} = 0,$$
$$\rho_{ae} = \rho_{ea}^{*}, \qquad \rho_{sg} = \rho_{gs}^{*} \quad (78)$$

at $t = 0$, these conditions still hold at any $t > 0$. The concurrence is determined by formula (76) with

$$C_1(t) = |\rho_{gs}(t)| - \sqrt{\rho_{ee}(t)\rho_{aa}(t)},$$
$$C_2(t) = |\rho_{ea}(t)| - \sqrt{\rho_{gg}(t)\rho_{ss}(t)}. \quad (79)$$

There is also another class of 'X'-states satisfying conditions

$$\rho_{ea} = \rho_{ae} = \rho_{eg} = \rho_{ge} = \rho_{gs} = \rho_{sg} = \rho_{as} = \rho_{sa} = 0,$$
$$\rho_{se} = \rho_{es}^{*}, \qquad \rho_{ag} = \rho_{ga}^{*} \quad (80)$$

at $t = 0$. These conditions still hold at any $t > 0$. In this case instead of formulae (77) we have

$$C_1(t) = |\rho_{ga}(t)| - \sqrt{\rho_{ee}(t)\rho_{ss}(t)},$$
$$C_2(t) = |\rho_{es}(t)| - \sqrt{\rho_{gg}(t)\rho_{aa}(t)}. \quad (81)$$

Using formulae (77), (79) or (81) and expressions (73a)–(73p) with given numerical values of decoherence parameters γ_1 and γ_2, we can study the time dependence of the concurrence (76) and obtain the following interesting result [22, 29, 30, 34]: for many states from the three above-mentioned classes (75), (78) and (80) with definite initial values of corresponding non-vanishing matrix elements, the concurrence $C(t)$ is positive in the interval $0 < t < t_1$ with some t_1, vanishes in the interval $t_1 < t < t_2$ with some t_2 and becomes positive again for some $t > t_2$. This means that two spin-qubits are entangled in the first interval, non-entangled in the second one and are entangled again after the t_2. At $t = t_1$ the sudden death of entanglement takes place, and at $t = t_2$ it revives.

5. Exactly solvable models of two coupled spin-qubits interacting with the environment

Analytical exact solutions of the rate equations of two-spin-qubit systems are particularly useful for studying the physical phenomena such as the QI exchange between two spin-qubits and the time evolution of the qubit–qubit entanglement, as was shown in the two preceding sections in the special cases of two coupled spin-qubits without interactions with the environment and two uncoupled spin-qubits interacting with the environment. Approximate numerical solutions of rate equations cannot clearly and completely demonstrate the above-mentioned phenomena. In general, however, systems of rate equations for two coupled spin-qubits interacting with the environment can be solved mainly by means of approximate numerical methods. Therefore it is worth investigating systems of two coupled spin-qubits interacting with the environment which have exactly solvable rate equations. In this section we review the results of the study on three exactly solvable models.

5.1. Two coupled spin-qubits in a spin-star environment

As a first example consider a system of two coupled spin-qubits in a spin-star environment consisting of the interacting nuclear spins in the thermodynamic limit at a finite temperature [6, 52]. All the spin–spin couplings are those of the XY Heisenberg exchange interactions. It was shown that in the limit of an infinite number of spins in the environment, the total Hamiltonian of the system under consideration is equivalent to that of the system consisting of two coupled spins and a quantum field of monoenergetic bosons with a interaction Hamiltonian of the Jaynes–Cummings type [53].

Introduce two sets of Pauli matrices σ_α and τ_α acting on spin indices of wavefunctions and density matrices of two spin-qubits called the σ-qubit or τ-qubit, respectively, $\alpha = x, y, z$ or $+, -, 3$. The system has the following total Hamiltonian

$$H = \frac{E}{2}(\sigma_3 + \tau_3) + J(\sigma_+\tau_- + \sigma_-\tau_+) + \omega_0 b^+ b + g[(\sigma_+ + \tau_+)b + (\sigma_- + \tau_-)b^+]. \tag{82}$$

Denote by $|ij, n\rangle, i, j = 1, 2; n = 0, 1, 2, \ldots$, the eigenstates of commuting operators σ_3, τ_3 and b^+b:

$$\sigma_3|1j, n\rangle = |1j, n\rangle, \qquad \sigma_3|2j, n\rangle = -|2j, n\rangle, \qquad j = 1, 2,$$
$$\tau_3|i1, n\rangle = |i1, n\rangle, \qquad \tau_3|i2, n\rangle = -|i2, n\rangle, \qquad i = 1, 2, \tag{83}$$
$$b^+b|ij, n\rangle = n|ij, n\rangle, \qquad i, j = 1, 2.$$

These orthogonal and normalized state vectors form the Fock basis in the Hilbert space of the state vectors of the system. This Hilbert space is the direct (infinite) sum of a one-dimensional subspace V_0 with basis vector $|22, 0\rangle$, a three-dimensional subspace V_1 with basis $|12, 0\rangle, |21, 0\rangle, |22, 1\rangle$ and an infinite number of four-dimension subspaces $V_n, n \geqslant 2$, with basis vectors $|11, n-2\rangle, |12, n-1\rangle, |21, n-1\rangle, |22, n\rangle$. Subspace V_0 contains only one eigenstate $|\psi^{(0)}\rangle$ of H with the corresponding eigenvalue $E^{(0)}$

$$H|\psi^{(0)}\rangle = E^{(0)}|\psi^{(0)}\rangle, \tag{84}$$

where

$$E^{(0)} = -E, \tag{85}$$

$$|\psi^{(0)}\rangle = |22, 0\rangle. \tag{86}$$

In subspace V_1 there are three eigenstates $|\psi_\xi^{(1)}\rangle$ of H with corresponding eigenvalues $E_\xi^{(1)}$:

$$H|\psi_\xi^{(1)}\rangle = E_\xi^{(1)}|\psi_\xi^{(1)}\rangle, \qquad \xi = 0, 1, 2, \tag{87}$$

where

$$E_0^{(1)} = -J, \qquad E_1^{(1)} = -\tfrac{1}{2}(E - \omega_0 - J) + \tfrac{1}{2}\Delta, \qquad E_2^{(1)} = -\tfrac{1}{2}(E - \omega_0 - J) - \tfrac{1}{2}\Delta, \tag{88}$$

$$\Delta = \sqrt{(E - \omega_0 + J)^2 + 8g^2}, \tag{89}$$

$$|\psi_0^{(1)}\rangle = \tfrac{1}{2}(|12, 0\rangle - |21, 0\rangle),$$
$$|\psi_i^{(1)}\rangle = A_i^{(1)}\frac{1}{\sqrt{2}}(|12, 0\rangle + |21, 0\rangle) + B_i^{(1)}|22, 1\rangle, \qquad i = 1, 2, \tag{90}$$

$$A_1^{(1)} = B_2^{(1)} = \frac{1}{2}\sqrt{1 + \frac{J + E - \omega_0}{\Delta}}, \qquad B_1^{(1)} = -A_2^{(1)} = \frac{1}{2}\sqrt{1 - \frac{J + E - \omega_0}{\Delta}}. \tag{91}$$

In each subspace $V_n, n \geqslant 2$, there are four eigenstates $|\psi_\xi^{(n)}\rangle$ of H with corresponding eigenvalues $E_\xi^{(n)}$:

$$H|\psi_\xi^{(n)}\rangle = E_\xi^{(n)}|\psi_\xi^{(n)}\rangle, \qquad n \geqslant 2, \quad \xi = 0, 1, 2, 3, \tag{92}$$

where

$$E_0^{(n)} = (n + 1)\omega_0 - J, \tag{93}$$

$E_i^{(n)}$, $i = 1, 2, 3$,are three roots of the rank 3 algebraic equation

$$[x - E - (n-2)\omega_0][x - J - (n-1)\,\omega_0][x + E - n\omega_0] - 2g^2(n-1)[x + E - n\omega_0] - 2g^2 n[x - E - (n-2)\omega_0] = 0, \tag{94}$$

and

$$|\psi_0^{(1)}\rangle = \tfrac{1}{2}(|12, n-1\rangle - |21, n-1\rangle),$$
$$|\psi_i^{(n)}\rangle = A_i^{(n)}\frac{1}{\sqrt{2}}(|12, n-1\rangle + |21, n-1\rangle) + C_i^{(n)}|22, n\rangle + B_i^{(1)}|11, n-2\rangle, \qquad i = 1, 2, 3, \tag{95}$$

$$A_i^{(n)} = \left\{1 + \frac{g^2 2(n-1)}{[E_i^{(n)} - E - (n-2)\omega_0]^2} + \frac{g^2 2n}{[E_i^{(n)} + E - n\omega_0]^2}\right\}^{-1/2},$$
$$B_i^{(n)} = \frac{g\sqrt{2(n-1)}}{E_i^{(n)} - E - (n-2)\omega_0}A_i^{(n)}, \tag{96}$$
$$C_i^{(n)} = \frac{g\sqrt{2n}}{E_i^{(n)} + E - n\omega_0}A_i^{(n)}.$$

In the resonance case $\omega_0 = E$ we have

$$
\begin{aligned}
&E_0^{(1)} = -J, \qquad E_1^{(1)} = \tfrac{1}{2}(J + \Delta^{\mathrm{res}}),\\
&E_2^{(1)} = \tfrac{1}{2}(J - \Delta^{\mathrm{res}}), \qquad E_1^{(n)} = (n-1)\omega_0,\\
&E_2^{(n)} = (n-1)\omega_0 + \frac{J + J_n}{2},\\
&E_3^{(n)} = (n-1)\omega_0 + \frac{J - J_n}{2},\\
&\Delta^{\mathrm{res}} = \sqrt{J^2 + 8g^2}, \qquad J_n = \sqrt{J^2 + 8(2n-1)g^2}.
\end{aligned}
\tag{97}
$$

Because the total Hamiltonian has eigenvalues and eigenstates which can be exactly calculated, this model is exactly solvable. Consider the elements of the total density matrix ρ_{tot} in the eigenstate basis

$$
\begin{aligned}
\rho^{00}(t) &= \langle \psi^{(0)} | \rho_{\mathrm{tot}}(0) | \psi^{(0)} \rangle,\\
\rho_\xi^{0m}(t) &= \langle \psi^{(0)} | \rho_{\mathrm{tot}}(0) | \psi_\xi^{(m)} \rangle,\\
\rho_\xi^{m0}(t) &= \langle \psi_\xi^{(m)} | \rho_{\mathrm{tot}}(0) | \psi^{(0)} \rangle,\\
\rho_{\xi\eta}^{mn}(t) &= \langle \psi_\xi^{(m)} | \rho_{\mathrm{tot}}(0) | \psi_\eta^{(n)} \rangle.
\end{aligned}
\tag{98}
$$

They have following exact expressions

$$
\begin{aligned}
&\rho^{00}(t) = \rho^{00}(0), \qquad \rho_\xi^{0m}(t) = \mathrm{e}^{\mathrm{i}E_\xi^{(m)}t}\rho_\xi^{0m}(0),\\
&\rho_\xi^{m0}(t) = \mathrm{e}^{-\mathrm{i}E_\xi^{(m)}t}\rho_\xi^{m0}(0),\\
&\rho_{\xi\eta}^{mn}(t) = \mathrm{e}^{-\mathrm{i}[E_\xi^{(m)} - E_\eta^{(n)}]t}\rho_{\xi\eta}^{mn}(0).
\end{aligned}
\tag{99}
$$

Performing a suitable unitary transformation we can derive exact analytical expressions for the elements of the total density matrix in the Fock basis

$$
\rho_{(ij,n)\,(kl,m)}(t) = \langle ij, n | \rho(t) | kl, m \rangle. \tag{100}
$$

Consider the reduced density matrix of the system of two spin-qubits

$$
\rho_{(ij)(kl)}(t) = \sum_{n=0}^{\infty} \rho_{(ij,n)(kl,n)}(t). \tag{101}
$$

In general each element $\rho_{(ij)(kl)}(t)$ at $t > 0$ depends on the initial values of 16 series of matrix elements of the form (100). However, if at the initial time $t = 0$ the quantum states of the two-spin-qubit system and the spin-star environment are independent and that of the latter is an equilibrium state at a given temperature, then initial values $\rho_{(ij,n)(kl,m)}(0)$ of all the elements of ρ in the Fock basis can be expressed in terms of the initial values of 16 elements of the reduced density matrix of the two-spin-qubit system defined in equation (101). In this case the time evolution of the reduced density matrix of this system can be represented in the form

$$
\rho_{(ij)(kl)}(t) = \sum_{i'j'k'l'} C_{(ij)(kl)}^{(i'j')(k'l')}(t)\rho_{(i'j')(k'l')}(0) \tag{102}
$$

with completely determined functions $C_{(ij)(kl)}^{(i'j')(k'l')}(t)$.

Besides the special cases with the validity of equation (102) for the reduced density matrix of two spin-qubits, there are special cases of another kind. In each special case of this kind only the matrix elements of ρ in an invariant subspace V_n with a definite positive integer n are non-vanishing. In V_1 we have a set of 9 matrix elements

$$
\begin{aligned}
&\rho_{11}^{(1)} = \rho_{(12,0)(12,0)}, \qquad \rho_{12}^{(1)} = \rho_{(12,0)(21,0)},\\
&\rho_{13}^{(1)} = \rho_{(12,0)(22,1)},\\
&\rho_{21}^{(1)} = \rho_{(21,0)(12,0)}, \qquad \rho_{22}^{(1)} = \rho_{(21,0)(21,0)},\\
&\rho_{23}^{(1)} = \rho_{(21,0)(22,1)},\\
&\rho_{31}^{(1)} = \rho_{(22,1)(12,0)}, \qquad \rho_{32}^{(1)} = \rho_{(22,1)(21,0)},\\
&\rho_{33}^{(1)} = \rho_{(22,1)(22,1)},
\end{aligned}
\tag{103}
$$

and in each V_n with $n \geqslant 2$ we have a set of 16 elements

$$
\begin{aligned}
&\rho_{11}^{(n)} = \rho_{(11,n-2)(11,n-2)}, \qquad \rho_{12}^{(n)} = \rho_{(11,n-2)(12,n-1)},\\
&\rho_{13}^{(n)} = \rho_{(11,n-2)(21,n-1)}, \qquad \rho_{14}^{(n)} = \rho_{(11,n-2)(22,n)},\\
&\rho_{21}^{(n)} = \rho_{(12,n-1)(11,n-2)}, \qquad \rho_{22}^{(n)} = \rho_{(12,n-1)(12,n-1)},\\
&\rho_{23}^{(n)} = \rho_{(12,n-1)(21,n-1)}, \qquad \rho_{24}^{(n)} = \rho_{(12,n-1)(22,n)},\\
&\rho_{31}^{(n)} = \rho_{(21,n-1)(11,n-2)}, \qquad \rho_{32}^{(n)} = \rho_{(21,n-1)(12,n-1)},\\
&\rho_{33}^{(n)} = \rho_{(21,n-1)(21,n-1)}, \qquad \rho_{34}^{(n)} = \rho_{(21,n-1)(22,n)},\\
&\rho_{41}^{(n)} = \rho_{(22,n)(11,n-2)}, \qquad \rho_{42}^{(n)} = \rho_{(22,n)(12,n-1)},\\
&\rho_{43}^{(n)} = \rho_{(22,n)(21,n-1)}, \qquad \rho_{44}^{(n)} = \rho_{(22,n)(22,n)}.
\end{aligned}
\tag{104}
$$

If at $t = 0$ all matrix elements not belonging to the set (103) vanish, then at $t > 0$ only matrix elements $\rho_{ab}^{(1)}(t)$, $a, b = 1, 2, 3$, of this set are non-vanishing. They have expressions of the form

$$
\rho_{ab}^{(1)}(t) = \sum_{p,q=1}^{3} R_{ab}^{pq}(t)\rho_{pq}^{(1)}(0) \tag{105}
$$

with explicitly calculated functions $R_{ab}^{pq}(t)$. Similarly, if at $t = 0$ all matrix elements not belonging to the set (104) with a definite number $n \geqslant 2$ vanish, then at $t > 0$ only matrix elements $\rho_{ab}^{(n)}(t)$, $a, b = 1, 2, 3, 4$, of this set are non-vanishing. They have expressions of the form

$$
\rho_{ab}^{(n)}(t) = \sum_{p,q=1}^{4} S_{ab}^{pq}(t)\rho_{pq}^{(n)}(0) \tag{106}
$$

with explicitly calculated functions $S_{ab}^{pq}(t)$.

5.2. Two coupled spin-qubits interacting with two separate environments

Consider a system of two identical interacting spin-qubits localized near one another in a constant magnetic field, and denote by $\sigma_\pm, \sigma_3$ and $\tau_\pm, \tau_3$ the components of the spin operators representing these spin-qubits. We assume that

Hamiltonian of this system without the interaction of the environment has the form

$$H_S = \frac{E}{2}(\sigma_3 + \tau_3) + J(\sigma_+\tau_- + \sigma_-\tau_+). \tag{107}$$

The Liouvillian superoperator L is expressed by the Lindblad formula. Suppose that the interaction of the environment on a spin-qubit does not depend on the presence of the other. Then, with the choice of $\sigma_+, \sigma_-, \sigma_3$ and τ_+, τ_-, τ_3 to be generators of the corresponding $SU(2)$ groups in the Lindblad formula, the Liouvillian superoperator has the form

$$L\rho = \sum_{i=\pm,3} \alpha_i \{\sigma_i\rho\sigma_i^+ - \tfrac{1}{2}\sigma_i^+\sigma_i\rho - \tfrac{1}{2}\rho\sigma_i^+\sigma_i + \tau_i\rho\tau_i^+ - \tfrac{1}{2}\tau_i^+\tau_i\rho - \tfrac{1}{2}\rho\tau_i^+\tau_i\}. \tag{108}$$

The constants α_-, α_+ and α_3 in equation (108) are the physical parameters characterizing the damping, the thermal excitation and the dephasing of each spin-qubit due to its interactions with the environment.

The system of rate equations was solved exactly, and analytical expressions of all 15 components $\rho_{(\alpha\beta)}(t)$, $(\alpha\beta) \neq (00)$, were derived in [54]. To simplify the text we present formulae in the special case, when the excitation and the dephasing are neglected and there is only the damping which plays the main role, by setting $\alpha_- = \gamma, \alpha_+ = \alpha_3 = 0$. In this case components $\rho_{(+0)}(t)$, $\rho_{(+3)}(t)$, $\rho_{(30)}(t)$ and $\rho_{(+-)}(t)$ have the following expressions:

$$\begin{aligned}\rho_{(+0)}(t) = \frac{1}{2}\mathrm{e}^{-\mathrm{i}Et}\mathrm{e}^{-\gamma t/2}\Bigg\{\Bigg[\left(\frac{J+\mathrm{i}\gamma}{2J+\mathrm{i}\gamma}\mathrm{e}^{-\mathrm{i}Jt} + \frac{J-\mathrm{i}\gamma}{2J-\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}Jt}\right)\\ + \left(\frac{J}{2J+\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}Jt} + \frac{J}{2J-\mathrm{i}\gamma}\mathrm{e}^{-\mathrm{i}Jt}\right)\mathrm{e}^{-\gamma t}\Bigg]\rho_{(+0)}(0)\\ + \Bigg[\left(\frac{J+\mathrm{i}\gamma}{2J+\mathrm{i}\gamma}\mathrm{e}^{-\mathrm{i}Jt} - \frac{J-\mathrm{i}\gamma}{2J-\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}Jt}\right)\\ + \left(\frac{J}{2J+\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}Jt} - \frac{J}{2J-\mathrm{i}\gamma}\mathrm{e}^{-\mathrm{i}Jt}\right)\mathrm{e}^{-\gamma t}\Bigg]\rho_{(0+)}(0)\\ - \Bigg[\left(\frac{J}{2J+\mathrm{i}\gamma}\mathrm{e}^{-\mathrm{i}Jt} + \frac{J}{2J-\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}Jt}\right)\\ - \left(\frac{J}{2J+\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}Jt} + \frac{J}{2J-\mathrm{i}\gamma}\mathrm{e}^{-\mathrm{i}Jt}\right)\mathrm{e}^{-\gamma t}\Bigg]\rho_{(+3)}(0)\\ - \Bigg[\left(\frac{J}{2J+\mathrm{i}\gamma}\mathrm{e}^{-\mathrm{i}Jt} - \frac{J}{2J-\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}Jt}\right)\\ - \left(\frac{J}{2J+\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}Jt} - \frac{J}{2J-\mathrm{i}\gamma}\mathrm{e}^{-\mathrm{i}Jt}\right)\mathrm{e}^{-\gamma t}\Bigg]\rho_{(3+)}(0)\Bigg\},\end{aligned} \tag{109}$$

$$\begin{aligned}\rho_{(+3)}(t) = \frac{1}{2}\mathrm{e}^{-\mathrm{i}Et}\mathrm{e}^{-\gamma t}\Bigg\{\Bigg[\left(\frac{J}{2J+\mathrm{i}\gamma}\mathrm{e}^{-\mathrm{i}Jt} + \frac{J}{2J-\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}Jt}\right)\\ + \left(\frac{J+\mathrm{i}\gamma}{2J+\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}Jt} + \frac{J-\mathrm{i}\gamma}{2J-\mathrm{i}\gamma}\mathrm{e}^{-\mathrm{i}Jt}\right)\mathrm{e}^{-\gamma t}\Bigg]\rho_{(+3)}(0)\\ + \Bigg[\left(\frac{J}{2J+\mathrm{i}\gamma}\mathrm{e}^{-\mathrm{i}Jt} - \frac{J}{2J-\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}Jt}\right)\\ + \left(\frac{J+\mathrm{i}\gamma}{2J+\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}Jt} - \frac{J-\mathrm{i}\gamma}{2J-\mathrm{i}\gamma}\mathrm{e}^{-\mathrm{i}Jt}\right)\mathrm{e}^{-\gamma t}\Bigg]\rho_{(3+)}(0)\\ - \Bigg[\left(\frac{J+\mathrm{i}\gamma}{2J+\mathrm{i}\gamma}\mathrm{e}^{-\mathrm{i}Jt} + \frac{J-\mathrm{i}\gamma}{2J-\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}Jt}\right)\\ - \left(\frac{J+\mathrm{i}\gamma}{2J+\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}Jt} + \frac{J-\mathrm{i}\gamma}{2J-\mathrm{i}\gamma}\mathrm{e}^{-\mathrm{i}Jt}\right)\mathrm{e}^{-\gamma t}\Bigg]\rho_{(+0)}(0)\\ - \Bigg[\left(\frac{J+\mathrm{i}\gamma}{2J+\mathrm{i}\gamma}\mathrm{e}^{-\mathrm{i}Jt} - \frac{J-\mathrm{i}\gamma}{2J-\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}Jt}\right)\\ - \left(\frac{J+\mathrm{i}\gamma}{2J+\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}Jt} - \frac{J-\mathrm{i}\gamma}{2J-\mathrm{i}\gamma}\mathrm{e}^{-\mathrm{i}Jt}\right)\mathrm{e}^{-\gamma t}\Bigg]\rho_{(0+)}(0)\Bigg\},\end{aligned} \tag{110}$$

$$\begin{aligned}\rho_{(30)}(t) = \tfrac{1}{2}\mathrm{e}^{-\gamma t}\{(1+\cos 2Jt)\rho_{(30)}(0)\\ + (1-\cos 2Jt)\rho_{(03)}(0) + \frac{\mathrm{i}}{2}\sin 2Jt\\ \times[\rho_{(+-)}(0) - \rho_{(-+)}(0)]\} - \tfrac{1}{4}(1-\mathrm{e}^{-\gamma t}),\end{aligned} \tag{111}$$

$$\begin{aligned}\rho_{(+-)}(t) = \tfrac{1}{2}\mathrm{e}^{-\gamma t}\{(1+\cos 2Jt)\rho_{(+-)}(0)\\ + (1-\cos 2Jt)\rho_{(-+)}(0) + 2\mathrm{i}\sin 2Jt\\ \times[\rho_{(30)}(0) - \rho_{(03)}(0)]\}.\end{aligned} \tag{112}$$

Components $\rho_{(-0)}(t)$ and $\rho_{(-3)}(t)$ have expressions obtained from equations (109) and (110) by a change of the sign of E and J: $E \to -E, J \to -J$, and the substitutions $\rho_{(+0)}(0) \to \rho_{(-0)}(0), \rho_{(0+)}(0) \to \rho_{(0-)}(0), \rho_{(+3)}(0) \to \rho_{(-3)}(0), \rho_{(3+)}(0) \to \rho_{(3-)}(0)$. Formulae of $\rho_{(0+)}(t), \rho_{(3+)}(t), \rho_{(0-)}(t), \rho_{(3-)}(t), \rho_{(03)}(t)$ and $\rho_{(-+)}(t)$ are obtained from expressions of $\rho_{(+0)}(t), \rho_{(+3)}(t), \rho_{(-0)}(t), \rho_{(-3)}(t), \rho_{(30)}(t)$ and $\rho_{(+-)}(t)$ by the interchanges $\rho_{(\pm 0)}(0) \leftrightarrow \rho_{(0\pm)}(0)$, $\rho_{(\pm 3)}(0) \leftrightarrow \rho_{(3\pm)}(0)$, $\rho_{(30)}(0) \leftrightarrow \rho_{(03)}(0)$ and $\rho_{(+-)}(0) \leftrightarrow \rho_{(-+)}(0)$. For $\rho_{(++)}(t), \rho_{(--)}(t)$ and $\rho_{(33)}(t)$ we have following expressions:

$$\rho_{(++)}(t) = \mathrm{e}^{-2\mathrm{i}Et}\mathrm{e}^{-\gamma t}\,\rho_{(++)}(0), \tag{113}$$

$$\rho_{(--)}(t) = \mathrm{e}^{2\mathrm{i}Et}\mathrm{e}^{-\gamma t}\,\rho_{(--)}(0). \tag{114}$$

$$\begin{aligned}\rho_{(33)}(t) = \mathrm{e}^{-2\gamma t}\rho_{(33)}(0) - \mathrm{e}^{-\gamma t}(1-\mathrm{e}^{-\gamma t})\\ \times[\rho_{(30)}(0) + \rho_{(03)}(0)] + \tfrac{1}{4}(1-\mathrm{e}^{-\gamma t})^2.\end{aligned} \tag{115}$$

Expressions of $\rho_{(\alpha 0)}(t) = S_\alpha(t)/2$ and $\rho_{(0\alpha)}(t) = T_\alpha(t)/2, \alpha = \pm, 3$, determine the reduced density matrices $\rho^\sigma(t)$ and $\rho^\tau(t)$ of two spin-qubits in the presence of their coupling and the interaction with the environment. Suppose that at the initial time $t = 0$ the quantum states of two spin-qubits are independent. Then these expressions for $S_\alpha(t)$ and $T_\alpha(t)$ demonstrate the mutual dependence of the reduced density matrices of two spin-qubits: $S_\pm(t)$ depends not only on $S_\pm(0)$, but also on $S_3(0)$ and $T_\pm(0)$; $S_3(t)$ depends not only on $S_3(0)$, but also on $T_3(0)$, $S_\pm(0)$ and $T_\pm(0)$, and vice versa. They exhibit the quantum information transfer from a spin-qubit to the second one due to their coupling in the presence of decoherence.

5.3. Two coupled spin-qubits interacting with a common environment

In the case of the interaction of both spin-qubits with one and the same environment besides the individual damping, excitation and dephasing of each spin-qubit there also take place the simultaneous collective damping, excitation and dephasing of both spin-qubits. Instead of equation (108) for the Liouvillian superoperator we now have the following formula

$$L\rho = \sum_{i=\pm,3} \alpha_i\{\sigma_i\rho\sigma_i^+ - \tfrac{1}{2}\sigma_i^+\sigma_i\rho - \tfrac{1}{2}\rho\sigma_i^+\sigma_i + \tau_i\rho\tau_i^+ - \tfrac{1}{2}\tau_i^+\tau_i\rho - \tfrac{1}{2}\rho\tau_i^+\tau_i\} + \sum_{i=\pm,3}\alpha_i'\{\sigma_i\rho\tau_i^+ - \tfrac{1}{2}\tau_i^+\sigma_i\rho - \tfrac{1}{2}\rho\tau_i^+\sigma_i + \tau_i\rho\sigma_i^+ - \tfrac{1}{2}\sigma_i^+\tau_i\rho - \tfrac{1}{2}\rho\sigma_i^+\tau_i\},$$
$$\alpha_i \geqslant \alpha_i' \geqslant 0. \tag{116}$$

To simplify the text we consider only the dampings (individual and collective) which play the main role, and set $\alpha_- = \gamma, \alpha_-' = \chi, \chi \leqslant \gamma, \alpha_+ = \alpha_+' = \alpha_3 = \alpha_3' = 0$. In this case the rate equations have the following exact solution:

$$\begin{aligned}
\rho_{(+0)}(t) &= \frac{1}{2}\mathrm{e}^{-\mathrm{i}Et}\left\{\frac{\mathrm{e}^{-(\gamma+\chi)t/2}}{2J+\mathrm{i}\gamma}\left[\left(J+\mathrm{i}\frac{\chi}{2}+\mathrm{i}\gamma\right)\mathrm{e}^{-\mathrm{i}Jt} + \left(J-\mathrm{i}\frac{\chi}{2}\right)\mathrm{e}^{\mathrm{i}Jt}\mathrm{e}^{-\gamma t}\right] + \frac{\mathrm{e}^{-(\gamma-\chi)t/2}}{2J-\mathrm{i}\gamma}\right.\\
&\quad\left.\times\left[\left(J+\mathrm{i}\frac{\chi}{2}-\mathrm{i}\gamma\right)\mathrm{e}^{\mathrm{i}Jt} + \left(J-\mathrm{i}\frac{\chi}{2}\right)\mathrm{e}^{-\mathrm{i}Jt}\mathrm{e}^{-\gamma t}\right]\right\}\\
&\quad\times\rho_{(+0)}(0) + \frac{1}{2}\mathrm{e}^{-\mathrm{i}Et}\left\{\frac{\mathrm{e}^{-(\gamma+\chi)t/2}}{2J+\mathrm{i}\gamma}\right.\\
&\quad\times\left[\left(J+\mathrm{i}\frac{\chi}{2}+\mathrm{i}\gamma\right)\mathrm{e}^{-\mathrm{i}Jt} + \left(J-\mathrm{i}\frac{\chi}{2}\right)\mathrm{e}^{\mathrm{i}Jt}\mathrm{e}^{-\gamma t}\right]\\
&\quad-\frac{\mathrm{e}^{-(\gamma-\chi)t/2}}{2J-\mathrm{i}\gamma}\left[\left(J+\mathrm{i}\frac{\chi}{2}-\mathrm{i}\gamma\right)\mathrm{e}^{\mathrm{i}Jt}\right.\\
&\quad\left.\left.+\left(J-\mathrm{i}\frac{\chi}{2}\right)\mathrm{e}^{-\mathrm{i}Jt}\mathrm{e}^{-\gamma t}\right]\right\}\rho_{(0+)}(0)\\
&\quad-\frac{1}{2}\mathrm{e}^{-\mathrm{i}Et}\left(J-\mathrm{i}\frac{\chi}{2}\right)\left\{\frac{\mathrm{e}^{-(\gamma+\chi)t/2}}{2J+\mathrm{i}\gamma}[\mathrm{e}^{-\mathrm{i}Jt}-\mathrm{e}^{\mathrm{i}Jt}\mathrm{e}^{-\gamma t}]\right.\\
&\quad\left.+\frac{\mathrm{e}^{-(\gamma-\chi)t/2}}{2J-\mathrm{i}\gamma}[\mathrm{e}^{\mathrm{i}Jt}-\mathrm{e}^{-\mathrm{i}Jt}\mathrm{e}^{-\gamma t}]\right\}\rho_{(+3)}(0)\\
&\quad-\frac{1}{2}\mathrm{e}^{-\mathrm{i}Et}\left(J-\mathrm{i}\frac{\chi}{2}\right)\left\{\frac{\mathrm{e}^{-(\gamma+\chi)t/2}}{2J+\mathrm{i}\gamma}[\mathrm{e}^{-\mathrm{i}Jt}-\mathrm{e}^{\mathrm{i}Jt}\mathrm{e}^{-\gamma t}]\right.\\
&\quad\left.-\frac{\mathrm{e}^{-(\gamma-\chi)t/2}}{2J-\mathrm{i}\gamma}[\mathrm{e}^{\mathrm{i}Jt}-\mathrm{e}^{-\mathrm{i}Jt}\mathrm{e}^{-\gamma t}]\right\}\rho_{(3+)}(0),
\end{aligned}\tag{117}$$

$$\begin{aligned}
\rho_{(+3)}(t) &= \frac{1}{2}\mathrm{e}^{-\mathrm{i}Et}\left\{\frac{\mathrm{e}^{-(\gamma+\chi)t/2}}{2J+\mathrm{i}\gamma}\left[\left(J-\mathrm{i}\frac{\chi}{2}\right)\mathrm{e}^{-\mathrm{i}Jt} + \left(J+\mathrm{i}\frac{\chi}{2}+\mathrm{i}\gamma\right)\mathrm{e}^{\mathrm{i}Jt}\mathrm{e}^{-\gamma t}\right] + \frac{\mathrm{e}^{-(\gamma-\chi)t/2}}{2J-\mathrm{i}\gamma}\right.\\
&\quad\left.\times\left[\left(J-\mathrm{i}\frac{\chi}{2}\right)\mathrm{e}^{\mathrm{i}Jt} + \left(J+\mathrm{i}\frac{\chi}{2}-\mathrm{i}\gamma\right)\mathrm{e}^{-\mathrm{i}Jt}\mathrm{e}^{-\gamma t}\right]\right\}\\
&\quad\times\rho_{(+3)}(0) + \frac{1}{2}\mathrm{e}^{-\mathrm{i}Et}\left\{\frac{\mathrm{e}^{-(\gamma+\chi)t/2}}{2J+\mathrm{i}\gamma}\right.\\
&\quad\times\left[\left(J-\mathrm{i}\frac{\chi}{2}\right)\mathrm{e}^{-\mathrm{i}Jt} + \left(J+\mathrm{i}\frac{\chi}{2}+\mathrm{i}\gamma\right)\mathrm{e}^{\mathrm{i}Jt}\mathrm{e}^{-\gamma t}\right]\\
&\quad-\frac{\mathrm{e}^{-(\gamma-\chi)t/2}}{2J-\mathrm{i}\gamma}\left[\left(J-\mathrm{i}\frac{\chi}{2}\right)\mathrm{e}^{\mathrm{i}Jt} + \left(J+\mathrm{i}\frac{\chi}{2}-\mathrm{i}\gamma\right)\right.\\
&\quad\left.\left.\times\,\mathrm{e}^{-iJt}\mathrm{e}^{-\gamma t}\right]\right\}\rho_{(3+)}(0) - \frac{1}{2}\mathrm{e}^{-\mathrm{i}Et}\left\{\frac{\mathrm{e}^{-(\gamma+\chi)t/2}}{2J+\mathrm{i}\gamma}\right.\\
&\quad\times\left(J+\mathrm{i}\frac{\chi}{2}+\mathrm{i}\gamma\right)[\mathrm{e}^{-\mathrm{i}Jt}-\mathrm{e}^{\mathrm{i}Jt}\mathrm{e}^{-\gamma t}]\\
&\quad+\frac{\mathrm{e}^{-(\gamma-\chi)t/2}}{2J-\mathrm{i}\gamma}\left(J+\mathrm{i}\frac{\chi}{2}-\mathrm{i}\gamma\right)[\mathrm{e}^{\mathrm{i}Jt}-\mathrm{e}^{-iJt}\mathrm{e}^{-\gamma t}]\Bigg\}\\
&\quad\times\rho_{(+0)}(0) - \frac{1}{2}\mathrm{e}^{-\mathrm{i}Et}\left\{\frac{\mathrm{e}^{-(\gamma+\chi)t/2}}{2J+\mathrm{i}\gamma}\left(J+\mathrm{i}\frac{\chi}{2}+\mathrm{i}\gamma\right)\right.\\
&\quad\times[\mathrm{e}^{-\mathrm{i}Jt}-\mathrm{e}^{\mathrm{i}Jt}\mathrm{e}^{-\gamma t}] - \frac{\mathrm{e}^{-(\gamma-\chi)t/2}}{2J-\mathrm{i}\gamma}\left(J+\mathrm{i}\frac{\chi}{2}-\mathrm{i}\gamma\right)\\
&\quad\left.\times[\mathrm{e}^{\mathrm{i}Jt}-\mathrm{e}^{-\mathrm{i}Jt}\mathrm{e}^{-\gamma t}]\right\}\rho_{(0+)}(0),
\end{aligned}\tag{118}$$

$\rho_{(-0)}(t)$ and $\rho_{(-3)}(t)$ have similar expressions with the change of the signs of E and J, $E \to -E, J \to -J$, and the substitutions $\rho_{(+0)}(0) \to \rho_{(-0)}(0), \rho_{(0+)}(0) \to \rho_{(0-)}(0), \rho_{(+3)}(0) \to \rho_{(-3)}(0), \rho_{(3+)}(0) \to \rho_{(3-)}(0)$,

$$\begin{aligned}
\rho_{(30)}(t) &= \frac{1}{4}\left[\frac{\gamma-\chi}{\gamma+\chi}\mathrm{e}^{-(\gamma-\chi)t} + \frac{\gamma+\chi}{\gamma-\chi}\mathrm{e}^{-(\gamma+\chi)t} - 4\frac{\chi^2}{\gamma^2-\chi^2}\mathrm{e}^{-2\gamma t} + 2\mathrm{e}^{-\gamma t}\cos 2Jt\right]\rho_{(30)}(0)\\
&\quad+\frac{1}{4}\left[\frac{\gamma-\chi}{\gamma+\chi}\mathrm{e}^{-(\gamma-\chi)t} + \frac{\gamma+\chi}{\gamma-\chi}\mathrm{e}^{-(\gamma+\chi)t} - 4\frac{\chi^2}{\gamma^2-\chi^2}\mathrm{e}^{-2\gamma t} - 2\mathrm{e}^{-\gamma t}\cos 2Jt\right]\rho_{(03)}(0)\\
&\quad-\tfrac{1}{8}[\mathrm{e}^{-(\gamma-\chi)t}-\mathrm{e}^{-(\gamma+\chi)t}-2\mathrm{i}\mathrm{e}^{-\gamma t}\sin 2Jt]\rho_{(+-)}(0)\\
&\quad-\tfrac{1}{8}[\mathrm{e}^{-(\gamma-\chi)t}-\mathrm{e}^{-(\gamma+\chi)t}+2\mathrm{i}\mathrm{e}^{-\gamma t}\sin 2Jt]\rho_{(-+)}(0)\\
&\quad-\frac{1}{2}\left[\frac{\chi}{\gamma+\chi}\mathrm{e}^{-(\gamma-\chi)t} - \frac{\chi}{\gamma-\chi}\mathrm{e}^{-(\gamma+\chi)t} + 2\frac{\chi^2}{\gamma^2-\chi^2}\mathrm{e}^{-2\gamma t}\right]\rho_{(33)}(0) + \frac{1}{8}\left[\frac{\gamma}{\gamma+\chi}\mathrm{e}^{-(\gamma-\chi)t}\right.\\
&\quad\left.+\frac{\gamma}{\gamma-\chi}\mathrm{e}^{-(\gamma+\chi)t} - 2\frac{\chi^2}{\gamma^2-\chi^2}\mathrm{e}^{-2\gamma t} - 2\right],
\end{aligned}\tag{119}$$

$$\begin{aligned}
\rho_{(+-)}(t) &= \tfrac{1}{4}[\mathrm{e}^{-(\gamma-\chi)t}+\mathrm{e}^{-(\gamma+\chi)t}+2\mathrm{e}^{-\gamma t}\cos 2Jt]\rho_{(+-)}(0)\\
&\quad+\tfrac{1}{4}[\mathrm{e}^{-(\gamma-\chi)t}+\mathrm{e}^{-(\gamma+\chi)t}-2\mathrm{e}^{-\gamma t}\cos 2Jt]\rho_{(-+)}(0)\\
&\quad-\frac{1}{2}\left[\frac{\gamma-\chi}{\gamma+\chi}\mathrm{e}^{-(\gamma-\chi)t} - \frac{\gamma+\chi}{\gamma-\chi}\mathrm{e}^{-(\gamma+\chi)t} + 4\frac{\chi\gamma}{\gamma^2-\chi^2}\mathrm{e}^{-2\gamma t} - 2\mathrm{i}\mathrm{e}^{-\gamma t}\sin 2Jt\right]\rho_{(30)}(0)\\
&\quad-\frac{1}{2}\left[\frac{\gamma-\chi}{\gamma+\chi}\mathrm{e}^{-(\gamma-\chi)t} - \frac{\gamma+\chi}{\gamma-\chi}\mathrm{e}^{-(\gamma+\chi)t} + 4\frac{\chi\gamma}{\gamma^2-\chi^2}\mathrm{e}^{-2\gamma t} + 2\mathrm{i}\mathrm{e}^{-\gamma t}\sin 2Jt\right]\rho_{(03)}(0)\\
&\quad+\left[\frac{\chi}{\gamma+\chi}\mathrm{e}^{-(\gamma-\chi)t} + \frac{\chi}{\gamma-\chi}\mathrm{e}^{-(\gamma+\chi)t} - 2\frac{\chi\gamma}{\gamma^2-\chi^2}\mathrm{e}^{-2\gamma t}\right]\rho_{33}(0) - \frac{1}{4}\left[\frac{\gamma}{\gamma+\chi}\mathrm{e}^{-(\gamma-\chi)t}\right.\\
&\quad\left.-\frac{\gamma}{\gamma-\chi}\mathrm{e}^{-(\gamma+\chi)t} + 2\frac{\chi\gamma}{\gamma^2-\chi^2}\mathrm{e}^{-2\gamma t}\right],
\end{aligned}\tag{120}$$

$\rho_{(0+)}(t), \rho_{(3+)}(t), \rho_{(0-)}(t), \rho_{(3-)}(t), \rho_{(03)}(t)$ and $\rho_{(-+)}(t)$ are obtained from the expression of $\rho_{(+0)}(t)$, $\rho_{(+3)}(t)$, $\rho_{(-0)}(t)$, $\rho_{(-3)}(t)$, $\rho_{(30)}(t)$ and $\rho_{(+-)}(t)$, respectively, by the interchanges $\rho_{(\pm3)}(0) \leftrightarrow \rho_{(3\pm)}(0), \rho_{(30)}(0) \leftrightarrow \rho_{(03)}(0)$ and $\rho_{(-+)}(0) \leftrightarrow \rho_{(+-)}(0)$,

$$\rho_{(++)}(t) = \mathrm{e}^{-\gamma t}\mathrm{e}^{-2\mathrm{i}Et}\rho_{(++)}(0), \tag{121}$$

$$\rho_{(--)}(t) = \mathrm{e}^{-\gamma t}\mathrm{e}^{2\mathrm{i}Et}\rho_{(--)}(0), \tag{122}$$

$$\rho_{(33)}(t) = \left[-\frac{1}{2}\frac{\gamma-\chi}{\gamma+\chi}e^{-(\gamma-\chi)t} - \frac{1}{2}\frac{\gamma+\chi}{\gamma-\chi}e^{-(\gamma+\chi)t} + \frac{\gamma^2+\chi^2}{\gamma^2-\chi^2}e^{-2\gamma t}\right][\rho_{(03)}(0)+\rho_{(30)}(0)] + \tfrac{1}{4}[e^{-(\gamma-\chi)t} - e^{-(\gamma+\chi)t}][\rho_{(+-)}(0)+\rho_{(-+)}(0)] + \left[\frac{\chi}{\gamma+\chi}e^{-(\gamma-\chi)t} - \frac{\chi}{\gamma-\chi}e^{-(\gamma+\chi)t} + \frac{\gamma^2+\chi^2}{\gamma^2-\chi^2}e^{-2\gamma t}\right]\rho_{(33)}(0) - \frac{1}{4}\left[\frac{\gamma}{\gamma+\chi}e^{-(\gamma-\chi)t} + \frac{\gamma}{\gamma-\chi}e^{-(\gamma+\chi)t} - \frac{\gamma^2+\chi^2}{\gamma^2-\chi^2}e^{-2\gamma t}\right] + \frac{1}{4}. \quad (123)$$

In the special case, when the collective damping is neglected and therefore $\chi = 0$, formulae (117)–(123) are reduced to the corresponding ones (109)–(115), respectively, for the system of two coupled identical spin-qubits interacting with two separate environments.

There is another interesting special case of two coupled identical spin-qubits with both individual and collective damping: that with $\chi = \gamma$. It takes place, for example, when the Hamiltonian of the interaction between the spin-qubit and the common environment has the following symmetrical expression

$$H_{\text{int}} = \sum_{\xi}[f_{\xi}^{*}(\sigma_- + \tau_-)a_{\xi}^{+} + f_{\xi}(\sigma_+ + \tau_+)a_{\xi}], \quad (124)$$

where a_ξ and a_ξ^+ are the destruction and creation operators of the bosonic excitations in the environment. In this case the reduced density matrix of the two-spin-qubit system has following asymptotic behavior in the limit $t \to \infty$: $\rho_{(30)}(t)$, $\rho_{(03)}(t)$, $\rho_{(+-)}(t)$, $\rho_{(-+)}(t)$, and $\rho_{(33)}(t)$ tend to limits which can be finite or vanishing, $\rho_{(++)}(t)$ and $\rho_{(--)}(t)$ tend to zero, but $\rho_{(\pm 0)}(t)$, $\rho_{(0\pm)}(t)$, $\rho_{(\pm 3)}(t)$ and $\rho_{(3\pm)}(t)$, are still coherently oscillating without damping, namely

$$\rho_{(\pm 0)}(t) \approx -\rho_{(0\pm)}(t) \approx -\rho_{(\pm 3)}(t) \approx \rho_{(3\pm)}(t) \approx \tfrac{1}{4}e^{\mp i(E-J)t}[\rho_{(\pm 0)}(0) - \rho_{(0\pm)}(0) - \rho_{(\pm 3)}(0) + \rho_{(3\pm)}(0)]. \quad (125)$$

The appearance of the asymptotic coherent oscillations (125) without damping signifies the existence of some asymptotically decoherence free subspace in the Hilbert space of the state vectors of the two-spin-qubit system. To affirm this statement it is convenient to write explicit analytical expressions for the elements of the reduced density matrix ρ in the Dicke basis, consider their asymptotic behavior at $t \to +\infty$ and obtain

$$\rho_{gg}(t) \approx 1 - \rho_{aa}(0), \qquad \rho_{aa}(t) \approx \rho_{aa}(0),$$
$$\rho_{ga}(t) \approx e^{i(E-J)t}\rho_{ga}(0), \qquad \rho_{ag}(t) \approx e^{-i(E-J)t}\rho_{ag}(0),$$
$$\rho_{ss}(t) \approx \rho_{ee}(t) \approx \rho_{es}(t) \approx \rho_{se}(t) \approx \rho_{eg}(t) \approx \rho_{ge}(t) \approx \rho_{sa}(t) \approx \rho_{as}(t) \approx \rho_{es}(t) \approx \rho_{se}(t) \approx \rho_{ea}(t) \approx \rho_{ae}(t) \approx 0. \quad (126)$$

The possible existence of decoherence free subspaces in Hilbert spaces of the state vectors of qubit systems has been discussed by many authors [55–59]. Here we have explicitly demonstrated the existence of an asymptotically decoherence free subspace of quantum states of two spin-qubit systems.

6. Conclusion

The results of a series of theoretical studies on quantum dynamics of two-spin-qubit systems have been presented in an unified style. The decoherence of the systems due to the interaction of spin-qubits with the environment was considered in the Markovian approximation. After presenting the general method for deriving the rate equations, different concrete models of two-spin-qubit systems were investigated. The rate equations of axially symmetric systems of two coupled spin-qubits, non-interacting or interacting with the environment, are exactly solvable, and analytical expressions of their exact solutions explicitly exhibit the physical mechanism of the QI exchange between two spin-qubits—the mutual dependence of QIs encoded into them. It was shown also how to apply the reasonings presented in the study of exactly solvable models to the investigation of two spin-qubits with more complicated rate equations, which can be solved only by means of approximate numerical methods. For an axially symmetric system of two spin-qubits interacting with a common environment its asymptotically decoherence free subspace was explicitly constructed.

In principle, solutions of rate equations of two spin-qubits can also be used for the study of qubit–qubit entanglement. In view of the appearance of a recent review on entanglement [60] we consider only one interesting physical phenomenon which was not mentioned in that review: entanglement sudden death and revival in the case of two uncoupled spin-qubits due to their interaction with the environment.

In recent works [61–68] an interest has arisen in the theoretical investigation of the quantum state transfer along spin-qubit chains. The calculation methods and physical reasonings presented in this topical review can be extended to the study of quantum state transfer not only along spin-qubit chains, but also in other many-spin-qubit systems.

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References

[1] Nielsen M A and Chang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
[2] Loss D and DiVincenzo D P 1998 *Phys. Rev.* A **57** 120
[3] Bagrov V G, Baldiotti M C, Gitman D M and Levin A D 2007 *Ann. Phys., Lpz.* **16** 274
[4] Trif M, Golovach V N and Loss D 2007 *Phys. Rev.* B **75** 085307
[5] Lucamarini M, Paganelli S and Mancini S 2004 *Phys. Rev.* A **69** 062308
[6] Yuan X Z, Goan H S and Zhu K D 2007 *Phys. Rev.* B **75** 045331

[7] Gunlycke D, Kendon V M, Vedral V and Bose S 2001 *Phys. Rev.* A **64** 042302
[8] Grigorenko I A and Khveshchenko D V 2005 *Phys. Rev. Lett.* **94** 040506
[9] Milburn G J, Laflamme R, Sanders B C and Knill E 2002 *Phys. Rev.* A **65** 032316
[10] Novaes M 2005 *Ann. Phys.* **318** 308
[11] Hamdouni Y 2007 *J. Phys. A: Math. Theor.* **40** 11569
[12] Storcz M J and Wilhelm F K 2003 *Phys. Rev.* A **67** 042319
[13] Storcz M J, Hellmann F, Hrelescu C and Wilhelm F K 2005 *Phys. Rev.* A **72** 052314
[14] Sinaysky I, Petruccione F and Burgarth D 2008 *Phys. Rev.* A **78** 062301
[15] Dubi Y and DiVentra M 2009 *Phys. Rev.* A **79** 012308
[16] Bandyopadhyay S and Lidar D A 2004 *Phys. Rev.* A **70** 010301(R)
[17] Rau A R P 2000 *Phys. Rev.* A **61** 032301
[18] Zhang J Z, Vala J, Sastry S and Whaley K B 2003 *Phys. Rev.* A **67** 042313
[19] Rau A R P, Selvaraj G and Uskov D 2005 *Phys. Rev.* A **71** 062316
[20] Majer J B, Paauw F G, ter Haar A C J, Harmans C J P M and Mooij J E 2005 *Phys. Rev. Lett.* **94** 090501
[21] Hartmann M J, Brandao F G S L and Plenio M B 2007 *Phys. Rev. Lett.* **99** 160501
[22] Ficek Z and Tanás R 2006 *Phys. Rev.* A **74** 024304
[23] Tanás R and Ficek Z 2004 *J. Opt. B: Quantum Semiclass. Opt.* **6** S90
Tanás R and Ficek Z 2004 *J. Opt. B: Quantum Semiclass. Opt.* **6** S610
[24] Ficek Z and Tanás R 2002 *Phys. Rep.* **372** 369
[25] Imamoglu A, Awschalom D D, Burkard G, DiVincenzo D P, Loss D, Sherwin M and Small A 1999 *Phys. Rev. Lett.* **83** 4204
[26] Yin Y and Evangelou S N 2008 *J. Phys. A: Math. Theor.* **41** 085306
[27] Kim M S, Lee J, Ahn D and Knight P L 2002 *Phys. Rev.* A **65** 040101
[28] Hamdouni Y, Fannes M and Petruccione F 2006 *Phys. Rev.* B **73** 245323
[29] Yu T and Eberly J H 2004 *Phys. Rev. Lett.* **93** 140404
Yu T and Eberly J H 2006 *Phys. Rev. Lett.* **97** 140403
[30] Yu T and Eberly J H 2007 *Quantum Inf. Comput.* **7** 459
[31] Jamróz A 2006 *J. Phys. A: Math. Gen.* **39** 7727
[32] Jakóbczyk L and Jamróz A 2004 *Phys. Lett.* A **333** 35
[33] Jakóbczyk L 2002 *J. Phys. A: Math. Gen.* **35** 6383
[34] Mazzola L, Maniscalco S, Piilo J, Suominen K A and Garraway B M 2009 *Phys. Rev.* A **79** 042302
[35] Cao X and Zheng H 2008 *Phys. Rev.* A **77** 022302
[36] Li Y, Zhou J and Guo H 2009 *Phys. Rev.* A **79** 012309
[37] Bellomo B, Franco R L and Compagno G 2007 *Phys. Rev. Lett.* **99** 160502
[38] Wang F Q, Zhang Z M and Liang R S 2008 *Phys. Rev.* A **78** 062318
[39] Dajku J, Mierzejewski M and Łuczka J 2008 *Phys. Rev.* A **77** 042316
[40] Maniscalco S and Petruccione F 2006 *Phys. Rev.* A **73** 012111
[41] Maniscalco S, Olivares S and Paris M G A 2007 *Phys. Rev.* A **75** 062119
[42] Chen P 2007 *Phys. Rev.* A **75** 062301
[43] Rebentrost P, Serban I, Schulte-Herbrüggen T and Wilhelm F K 2009 *Phys. Rev. Lett.* **102** 090401
[44] Weiss U 1999 *Quantum Dissipative Systems* 2nd edn (Singapore: World Scientific)
[45] Blum K 1981 *Density Matrix Theory and Application* (New York: Plenum)
[46] Bloch F 1956 *Phys. Rev.* **102** 104
Bloch F 1957 *Phys. Rev.* **105** 1206
[47] Redfield A G 1957 *IBM J. Res. Dev.* **1** 19
[48] Argyres P N and Kelley P K 1964 *Phys. Rev.* **134** A98
[49] Gorini V, Kossakowskii A and Sudarshan E C G 1976 *J. Math. Phys.* **17** 821
[50] Linblad G 1976 *Commun. Math. Phys.* **48** 119
[51] Dicke R H 1954 *Phys. Rev.* **93** 99
[52] Nguyen V H, Nguyen B H, Ho C S and Nguyen D V 2009 *Adv. Nat. Sci.* **10** 141
[53] Jaynes E T and Cummings F W 1963 *Proc. IEEE* **51** 89
[54] Nguyen V H, Nguyen B H and Le T H L 2008 *Proc. APCTP—ASEAN Workshop on Advanced Materials and Nanotechnology (Nha Trang, Sept.)* p 145
[55] Zanardi P and Rasetti M 1997 *Phys. Rev. Lett.* **79** 3306
[56] Duan L M and Gou G C 1998 *Phys. Rev.* A **57** 737
[57] Lidar D A, Chuang I L and Whaley K B 1998 *Phys. Rev. Lett.* **81** 2594
[58] Bacon D, Lidar D A and Whaley K B 1999 *Phys. Rev.* A **60** 1944
[59] Wu L A and Lidar D A 2002 *Phys. Rev. Lett.* **88** 207902
[60] Amico L, Fazio R, Osterloh A and Vedral V 2008 *Rev. Mod. Phys.* **80** 517
[61] Subrahmanyam V 2004 *Phys. Rev.* A **69** 034304
[62] Christandl M, Datta N, Ekert A and Landahl A J 2004 *Phys. Rev. Lett.* **92** 187902
[63] Qian X F, Li Y, Li Y, Song Z and Sun C P 2005 *Phys. Rev.* A **72** 062329
[64] Karbach P and Stolze J 2005 *Phys. Rev.* A **72** 030301(R)
[65] Yung M H 2006 *Phys. Rev.* A **74** 030303(R)
[66] Wójcik A, Łuczak T, Kurzynski P, Grudka A, Gdala T and Bednarska M 2007 *Phys. Rev.* A **75** 022330
[67] Zhang J, Rajendran N, Peng X and Suter D 2007 *Phys. Rev.* A **76** 012317
[68] Hao X and Zhu S 2008 *Phys. Rev.* A **78** 044302